International Journal of Theoretical Physics, Vol. 28, No. 8, 1989

Geometry of the Submanifolds of SEX_n. I. The C-Nonholonomic Frame of Reference

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Received February 27, 1989

A connection which is both Einstein and semisymmetric is called an SE connection. A generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through an SE connection is called an n-dimensional SE manifold and denoted by SEX_n. This paper is the introductory part of a systematic study of the submanifolds of SEX_n. It introduces a new concept of the C-nonholonomic frame of references in SEX_n at points of its submanifold and deals with its consequences. The second part will deal with the generalized fundamental equations on an SE hypersubmanifold of SEX_n. The third part will be devoted to the theory of parallelism in SEX_n and in its submanifold. Finally, the last part will study the curvature theory in a submanifold of SEX_n.

1. INTRODUCTION

In Appendix II to his last book, Einstein (1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. However, the geometrical consequences of these postulates were not developed very far by Einstein. Characterizing Einstein's 4-dimensional unified field theory as a set of geometrical postulates for X_4, Hlavatý (1957) gave its mathematical foundation for the first time. Since then the geometrical consequences of these postulates have been developed very far, mainly by Hlavatý. A number of mathematicians and theoretical physicists have contributed to the development of this theory.

Generalizing X_4 to an n-dimensional generalized Riemannian manifold X_n, Wrede (1958) studied Principles A and B of Einstein's unified field

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0020-7748/89/0800-0851$06.00/0

theory for the first time. But his solution of Einstein's equations is not surveyable. We also note that Hlavatý's solution for the first class in X_4 is not surveyable either. Later, Chung (1963) gave a very handy and surveyable solution in the 4-dimensional $^*g$-unified field theory using the method of substitution. Chung *et al.* (1981*a*,*b*, 1985) and Mishra (1962) also investigated the n-dimensional generalization of Principle A, using n-dimensional recurrence relations in X_n. However, they also failed to display a surveyable tensorial solution of Einstein's equations for the n-dimensional case, probably due to the complexity of the higher dimensions.

Recently Chung *et al.* (1987) introduced the concept of the n-dimensional SE manifold SEX_n, imposing a semisymmetric condition on X_n, and found a unique representation of Einstein's connection in a beautiful and surveyable form. Later, Chung *et al.* (1988*a*,*b*) also investigated curvature theory and field equations in SEX_n.

In a series of papers we shall establish a foundation of the geometry of submanifolds of SEX_n. The purpose of the present paper is to introduce a new concept of the C-nonholonomic frame of reference in a general X_n at points of its submanifold and to deal with its consequences in X_n and SEX_n. This paper contains four sections. Section 2 introduces some preliminary notations, concepts, and results which are needed in this and subsequent papers. Section 3 deals with the C-nonholonomic frame of reference and its consequences in a general X_n. The last section is devoted exclusively to the submanifolds of SEX_n, and in this section we prove the so-called "SE identity."

All considerations in the present paper are for a general $n>1$ and for all possible classes and indices of inertia.

2. PRELIMINARIES

This section is a brief collection of basic concepts, results, and notations which are needed in our subsequent considerations in the present paper.

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system y^ν, which obeys coordinate transformation $y^\nu \to \bar{y}^\nu$, for which

$$\mathrm{Det}\left(\frac{\partial \bar{y}}{\partial y}\right) \neq 0 \tag{2.1}$$

the manifold X_n is endowed with a general real, nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$,[4]

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

[4]Throughout the present paper, Greek indices are used for the holonomic components of tensors in X_n. They take the values $1^{\cdot}, 2^{\cdot}, \ldots, n^{\cdot}$ and follow the summation convention.

where

$$\mathfrak{g} = \mathrm{Det}(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = \mathrm{Det}(h_{\lambda\mu}) \neq 0 \tag{2.3}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^{\nu}_{\mu} \tag{2.4}$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of holonomic components of tensors in X_n in the usual manner.

The space X_n is assumed to be connected by a real, general connection $\Gamma^{\nu}_{\lambda\mu}$ with the following transformation rule:

$$\bar{\Gamma}^{\nu}_{\lambda\mu} = \frac{\partial \bar{y}^{\nu}}{\partial y^{\alpha}} \left(\frac{\partial y^{\beta}}{\partial \bar{y}^{\lambda}} \frac{\partial y^{\gamma}}{\partial \bar{y}^{\mu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 y^{\alpha}}{\partial \bar{y}^{\lambda} \partial \bar{y}^{\mu}} \right) \tag{2.5}$$

It may also be decomposed into its symmetric part $\Lambda^{\nu}_{\lambda\mu}$ and skew-symmetric part $S_{\lambda\mu}{}^{\nu}$:

$$\Gamma^{\nu}_{\lambda\mu} = \Lambda^{\nu}_{\lambda\mu} + S_{\lambda\mu}{}^{\nu} \tag{2.6}$$

where

$$\Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{(\lambda\mu)}, \qquad S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

Here $\Lambda^{\nu}_{\lambda\mu}$ is a connection and $S_{\lambda\mu}{}^{\nu}$ is a tensor, called the *torsion tensor* of the connection $\Gamma^{\nu}_{\lambda\mu}$.

The connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *Einstein* if it satisfies the following system of Einstein equations:

$$\partial_{\omega} g_{\lambda\mu} - \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} - \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} = 0 \tag{2.8a}$$

or equivalently

$$D_{\omega} g_{\lambda\mu} = 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.8b}$$

where D_{ω} denotes the symbol of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$. The manifold X_n connected by the Einstein connection will be denoted by EX_n. In fact, EX_n is a generalization of the space-time X_4. The connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *semisymmetric* if its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form

$$S_{\lambda\mu}{}^{\nu} = 2\delta_{[\lambda}{}^{\nu} X_{\mu]} \tag{2.9}$$

A connection $\Gamma^{\nu}_{\lambda\mu}$ which is both semisymmetric and Einstein is called an *SE-connection.* An *n-dimensional SE manifold, denoted by SEX_n* in further considerations, is a manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through an SE connection $\Gamma^{\nu}_{\lambda\mu}$.

A procedure similar to Christoffel's elimination applied to the symmetric part of (2.8b) yields that if the equations (2.8) admit a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form (Hlavatý, 1957)

$$\Gamma^{\nu}_{\lambda\mu} = \{^{\nu}_{\lambda\mu}\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} \tag{2.10}$$

where

$$U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} \tag{2.11}$$

and $\{^{\nu}_{\lambda\mu}\}$ are Christoffel symbols of $h_{\lambda\mu}$.

It has also been shown (Chung *et al.*, 1987) that there always exists a unique n-dimensional SE connection $\Gamma^{\nu}_{\lambda\mu}$ of the form

$$\Gamma^{\nu}_{\lambda\mu} = \{^{\nu}_{\lambda\mu}\} + 2k_{(\lambda}{}^{\nu} X_{\mu)} + 2\delta_{[\lambda}{}^{\nu} X_{\mu]} \tag{2.12}$$

for a unique vector X_λ given by

$$X_\lambda = \frac{1}{n-1} {}^{*}h^{\alpha\beta} \nabla_\alpha k_{\beta\lambda} \tag{2.13}$$

where ∇_λ is the symbolic vector of the covariant derivative with respect to $\{^{\nu}_{\lambda\mu}\}$. Therefore, we note that *there exists one and only one* SEX_n.

The following quantities will be used in our further considerations:

$$\mathfrak{k} = \mathrm{Det}(k_{\lambda\mu}) \tag{2.14}$$

$$g = \mathfrak{g}/\mathfrak{h}, \qquad k = \mathfrak{k}/\mathfrak{h} \tag{2.15}$$

$$^{(0)}k_{\lambda}{}^{\nu} = \delta_{\lambda}{}^{\nu}, \qquad ^{(p)}k_{\lambda}{}^{\nu} = {}^{(p-1)}k_{\lambda}{}^{\alpha} k_{\alpha}{}^{\nu}, \qquad p = 1, 2, \ldots \tag{2.16}$$

3. THE *C*-NONHOLONOMIC FRAME OF REFERENCE FOR A SUBMANIFOLD OF X_n

In this section we establish the foundation of the theory of submanifolds X_m of X_n, employing a new concept of *C-nonholonomic frame of reference in* X_n *at points of* $X_m (m < n)$, and derive some useful consequences of the frame.

Agreement 3.1. In our further considerations in the present and subsequent papers, we use the following different types of indices:

(a) Small Greek indices $\alpha, \beta, \gamma, \ldots$, running from 1 to n, and used for the holonomic components of tensors in X_n.

(b) Capital Roman italic indices $A, B, C, \ldots$, running from 1 to n and used for the C-nonholonomic components of tensors in X_n at points of X_m.

(c) Small Roman italic indices $i, j, k, \ldots$, with the exception of x, y, and z, running from 1 to m ($<n$).

(d) Small Roman italic indices x, y, and z running from $m+1$ to n.

The summation convention is operative with respect to each set of the above indices within their range, with the exception of x, y, and z.

3.1. *C*-Nonholonomic Frame of Reference

Let X_m be a subspace of a generalized n-dimensional Riemannian manifold X_n, defined by a system of real parametric equations

$$y^\nu = y^\nu(x^1, \ldots, x^m) \tag{3.1}$$

It is assumed that the functions $y^\nu(x^i)$ are sufficiently differentiable and the rank of the matrix of derivatives

$$B_i^\nu = \frac{\partial y^\nu}{\partial x^i} \tag{3.2}$$

is m. Clearly, the subspace X_m is an m-dimensional differentiable manifold in its own right.

The quantities B_i^ν are contravariant components of a vector in the y's and covariant components of a vector in the x's, respectively. In fact, they represent a vector in X_n tangential to the coordinate curves of parameter x^i of X_m. The vectors B_i^ν therefore form *the first set* of linearly independent vectors tangential to X_m and generate the tangent space $T_m(P)$ of X_m at a point P of X_m. Hence, any vector tangential to X_m must be expressible as a linear combination of the B_i^ν. In particular, if dy^ν is a small displacement vector tangential to X_m and dx^i denotes the same displacement vector in terms of the coordinates of X_m, we have

$$dy^\nu = B_i^\nu \, dx^i \tag{3.3}$$

More generally, if a vector field tangential to X_m is given by U^ν in the y's and U^i in the x's, respectively, we must have

$$U^\nu = B_i^\nu U^i \tag{3.4}$$

Since the rank of the matrix (B_i^ν) is m, the condition (2.3) guarantees the existence of *the first set* of $n-m$ nonnull real vectors $\underset{x}{N}{}^\nu$ normal to X_m, which are linearly independent and mutually orthogonal. That is,

$$h_{\alpha\beta} B_i^\alpha \underset{x}{N}{}^\beta = 0, \qquad h_{\alpha\beta} \underset{x}{N}{}^\alpha \underset{y}{N}{}^\beta = 0 \qquad \text{for} \quad x \neq y \tag{3.5a}$$

The process of determining this set is not unique unless $m = n - 1$. However, we may choose their magnitudes such that

$$h_{\alpha\beta} \underset{x}{N}^{\alpha} \underset{x}{N}^{\beta} = \varepsilon_x \tag{3.5b}$$

where $\varepsilon_x = \pm 1$ according as the left-hand side of (3.5b) is positive or negative.

Put

$$E_A^{\nu} = \begin{cases} B_i^{\nu} & \text{if} \quad A = 1, \ldots, m \quad (= i) \\ \underset{x}{N}^{\nu} & \text{if} \quad A = m+1, \ldots, n \quad (= x) \end{cases} \tag{3.6}$$

Since $\{E_A^{\nu}\}$ is a set of n linearly independent vectors in X_n at points of X_m, there exists a unique *second* set $\{E_{\lambda}^{A}\}$ of n linearly independent vectors at points of X_m such that

$$E_{\lambda}^{A} E_A^{\nu} = \delta_{\lambda}^{\nu}, \qquad E_{\alpha}^{A} E_B^{\alpha} = \delta_B^A \tag{3.7}$$

Now we are ready to introduce the following definition.

Definition 3.2. The sets E_A^{ν} and E_{λ}^{A} will be referred to as *the C-nonholonomic frame of reference* in X_n at points of X_m. This frame of reference gives rise to *C-nonholonomic components* of a tensor in X_n: If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor in X_n, then at points of X_m its C-nonholonomic components $T_{B\cdots}^{A\cdots}$ are defined by

$$T_{B\cdots}^{A\cdots} = T_{\beta\cdots}^{\alpha\cdots} E_{\alpha}^{A} \cdots E_B^{\beta} \cdots \tag{3.8}$$

Remark 3.3. In virtue of (3.7), an easy inspection shows that

$$T_{\lambda\cdots}^{\nu\cdots} = T_{B\cdots}^{A\cdots} E_A^{\nu} \cdots E_{\lambda}^{B} \cdots \tag{3.9}$$

Theorem 3.4. The C-nonholonomic components

$$h_{AB} = h_{\alpha\beta} E_A^{\alpha} E_B^{\beta}, \qquad h^{AB} = h^{\alpha\beta} E_{\alpha}^{A} E_{\beta}^{B} \tag{3.10}$$

are given by the matrix equations

$$(h_{AB}) = \left(\begin{array}{ccc:ccc} h_{11} & \cdots & h_{1m} & & & \\ \vdots & & \vdots & & 0 & \\ h_{m1} & \cdots & h_{mm} & & & \\ \hdashline & & & \varepsilon_{m+1} & & \\ & 0 & & & \ddots & \\ & & & & & \varepsilon_n \end{array} \right) \tag{3.11a}$$

$$
(h^{AB}) = \left(\begin{array}{ccc:ccc}
h^{11} & \cdots & h^{1m} & & & \\
\vdots & & \vdots & & 0 & \\
h^{m1} & \cdots & h^{mm} & & & \\
\hdashline
& & & \varepsilon_{m+1} & & \\
& 0 & & & \ddots & \\
& & & & & \varepsilon_n
\end{array}\right) \tag{3.11b}
$$

Proof. The properties (3.5) of vectors E_A^ν give rise to the matrix (h_{AB}). The components h^{AB} are obtained from

$$h_{AB}h^{AC} = \delta_B^C \tag{3.12}$$

which follows from (2.4), (3.7), and (3.10). ■

Theorem 3.5. In X_n the following relations hold:

$$E_A^\nu = E_\alpha^B h_{AB} h^{\nu\alpha}, \qquad E_\lambda^A = E_B^\alpha h^{AB} h_{\lambda\alpha} \tag{3.13}$$

Proof. Put

$$X_A^\nu = E_\alpha^B h_{AB} h^{\nu\alpha}$$

Then, in virtue of (2.4) and (3.9), we have

$$X_A^\nu E_\lambda^A = h_{AB} E_\lambda^A E_\alpha^B h^{\nu\alpha} = h_{\lambda\alpha} h^{\nu\alpha} = \delta_\lambda^\nu$$

Comparing this result with (3.7), we have $X_A^\nu = E_A^\nu$. The second equation may be obtained similarly.

Put

$$E_\lambda^A = \begin{cases} B_\lambda^i & \text{if} \quad A = 1, \ldots, m \quad (=i) \\ \overset{x}{N}_\lambda & \text{if} \quad A = m+1, \ldots, n \quad (=x) \end{cases} \tag{3.14}$$

$$B_\lambda^\nu = B_\lambda^i B_i^\nu \tag{3.15}$$

In the following theorem, we derive several useful relations in the theory of submanifolds of X_n.

Theorem 3.6. B_i^ν, B_λ^i, B_λ^ν, $\underset{x}{N}^\nu$, and $\overset{x}{N}_\lambda$ are tensors involved in the following identities:

$$B_\alpha^i B_j^\alpha = \delta_j^i, \qquad \overset{x}{N}_\alpha \underset{y}{N}^\alpha = \delta_y^x, \qquad B_\alpha^i \underset{x}{N}^\alpha = \overset{x}{N}_\alpha B_i^\alpha = 0 \tag{3.16}$$

$$B_\lambda^i = B_j^\alpha h_{\lambda\alpha} h^{ij}, \qquad \overset{x}{N}_\lambda = \varepsilon_x \underset{x}{N}_\lambda \tag{3.17a}$$

$$B^\nu_i = B^j_\alpha h^{\nu\alpha} h_{ij}, \qquad \underset{x}{N}^\nu = \varepsilon_x \overset{x}{N}^\nu \tag{3.17b}$$

$$h^{\nu\alpha} B^i_\alpha = h^{ij} B^\nu_j, \qquad h_{\lambda\alpha} B^\nu_i = h_{ij} B^j_\lambda \tag{3.18}$$

$$B^\nu_\lambda = \delta^\nu_\lambda - \sum_x \overset{x}{N}_\lambda \underset{x}{N}^\nu \tag{3.19a}$$

$$B^\alpha_\lambda \overset{x}{N}_\alpha = B^\nu_\alpha \underset{x}{N} = 0 \tag{3.19b}$$

$$B^\alpha_\lambda B^i_\alpha = B^i_\lambda, \qquad B^\nu_\alpha B^\alpha_i = B^\nu_i, \qquad B^\nu_\alpha B^\alpha_\lambda = B^\nu_\lambda \tag{3.19c}$$

Proof. The relations (3.16) and (3.19a) follow from (3.7) in virtue of (3.6), (3.14), and (3.15). The relations (3.17) are immediate consequences of (3.11) and (3.13). In fact, the relations (3.17b) are equivalent to those in (3.17a). The relations (3.18) may be obtained from (3.17). The relations satisfied by the tensor B^ν_λ may be proved easily by using (3.16) and (3.19a). ■

Remark 3.7. In virtue of (3.16), we note that the set $\{B^i_\lambda\}$ spans the dual space of the tangent space $T_m(P)$ of X_m at P. In fact, the vectors B^i_λ form *the second set* of linearly independent vectors tangential to X_m. On the other hand, we also note that the set $\{\overset{x}{N}_\lambda\}$ is *the second set* of $n-m$ nonnull real vectors normal to X_m, which are linearly independent and mutually orthogonal. They satisfy the last relations of (3.16) and (3.17). Obviously, the second set is uniquely determined if the first set is given.

Remark 3.8. As an application of the last relations of (3.17a) and (3.17b) we have

$$T^{xyz\cdots} = \varepsilon_x \varepsilon_y \varepsilon_z T_{xyz\cdots} \tag{3.20}$$

since

$$\begin{aligned} T^{xyz\cdots} &= T^{\alpha\beta\gamma\cdots} \overset{x}{N}_\alpha \overset{y}{N}_\beta \overset{z}{N}_\gamma \cdots \\ &= T_{\alpha\beta\gamma\cdots} \overset{x}{N}^\alpha \overset{y}{N}^\beta \overset{z}{N}^\gamma \cdots \\ &= \varepsilon_x \varepsilon_y \varepsilon_z \cdots T_{\alpha\beta\gamma\cdots} \underset{x}{N}^\alpha \underset{y}{N}^\beta \underset{z}{N}^\gamma \cdots \\ &= \varepsilon_x \varepsilon_y \varepsilon_z T_{xyz\cdots} \end{aligned}$$

3.2. Induced Tensors

Let

$$\bar{y}^\nu = \bar{y}^\nu(y^\mu) \tag{3.21a}$$

$$\tilde{x}^i = \tilde{x}^i(x^j) \tag{3.21b}$$

be two distinct types of coordinate transformations, the former on X_n and the latter on X_m. They are of class C^2 with nonvanishing Jacobians and are entirely independent of each other.

As stated at the beginning of the previous section, obviously the B_i^ν are components of a type (1, 0) tensor relative to (3.21a) and behave as components of a type (0, 1) tensor relative to (3.21b).

First of all, we have the following theorem as a consequence of (3.9).

Theorem 3.9. At each point of X_m any vector T^ν in X_n may be expressed as the sum of two vectors $T^i B_i^\nu$ and $\sum_x T^x \underset{x}{N}^\nu$, the former tangential to X_m, the latter normal to X_m. That is,

$$T^\nu = T^i B_i^\nu + \sum_x T^x \underset{x}{N}^\nu \tag{3.22a}$$

or equivalently

$$T_\lambda = T_i B_\lambda^i + \sum_x T_x \overset{x}{N}_\lambda \tag{3.22b}$$

Furthermore, T^i are components of a tangent vector relative to (3.21b), while T^x is invariant relative to (3.21a) and (3.21b).

Proof. The relation (3.22a) is another form of (3.9). The equivalence of (3.22a) to (3.22b) follows by multiplying by $h_{\lambda\nu}$ on both sides of (3.22a) and using (3.17b) and (3.20). It is clear that the first terms of (3.22a) and (3.22b) are tangential parts and the second terms normal parts of T^ν and T_λ, respectively. On the other hand, in virtue of (3.2), (3.17b), and (3.21b), we have

$$\begin{aligned}\tilde{B}_\alpha^i = \tilde{B}_j^\beta \tilde{h}^{ij} \tilde{h}_{\alpha\beta} &= \left(\frac{\partial y^\beta}{\partial x^k}\frac{\partial x^k}{\partial \tilde{x}^j}\right)\left(h^{pq}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial \tilde{x}^j}{\partial x^q}\right) h_{\alpha\beta} \\ &= B_k^\beta h_{\alpha\beta} h^{pk}\frac{\partial \tilde{x}^i}{\partial x^p} = B_\alpha^p \frac{\partial \tilde{x}^i}{\partial x^p}\end{aligned}$$

from which it follows that

$$\tilde{T}^i = \tilde{T}^\alpha \tilde{B}_\alpha^i = T^\alpha B_\alpha^k \frac{\partial \tilde{x}^i}{\partial x^k} = T^k \frac{\partial \tilde{x}^i}{\partial x^k} \tag{3.23}$$

The relation (3.23) shows that the T^i are components of a tangent vector to X_m relative to (3.21b). The proof of the last assertions is obvious. ■

In the previous theorem, we have seen that, if T^ν are components in the y's of a vector in X_n, then the vector $T^i B_i^\nu$ is its tangential part to X_m in the y's (while T^A are its C-nonholonomic components) at points of X_m, and T^i are components of a tangent vector to X_m at points of X_m relative to (3.21b). We call T^i *the induced vector on* X_m *of* T^ν *in* X_n. In the following definition we generalize this concept to a general tensor $T_{\lambda\cdots}^{\nu\cdots}$ in like manner.

Definition 3.10. If $T_{\lambda\cdots}^{\nu\cdots}$ are the components in the y's of a tensor in X_n, the quantities

$$T_{j\cdots}^{i\cdots} = T_{\beta\cdots}^{\alpha\cdots} B_\alpha^i \cdots B_j^\beta \cdots \tag{3.24}$$

evaluated at points of X_m, are components of a tensor in X_m relative to (3.21b) and are called *the components of the induced tensor on* X_m of $T_{\lambda\cdots}^{\nu\cdots}$ in X_n.

Therefore, the induced metric tensor g_{ij} on X_m of $g_{\lambda\mu}$ in X_n may be given by[5]

$$g_{ij} = g_{\alpha\beta} B_i^\alpha B_j^\beta \tag{3.25}$$

where its symmetric part h_{ij} and skew-symmetric part k_{ij} are

$$h_{ij} = h_{\alpha\beta} B_i^\alpha B_j^\beta, \qquad k_{ij} = k_{\alpha\beta} B_i^\alpha B_j^\beta \tag{3.26}$$

so that

$$g_{ij} = h_{ij} + k_{ij} \tag{3.27}$$

In the present paper, *we restrict ourselves to subspaces for which the following condition holds*[6]:

$$\mathrm{Det}(h_{ij}) \neq 0 \tag{3.28}$$

In virtue of the condition (3.28), we may define a unique tensor $\bar{h}^{ij}$ by

$$h_{ij}\bar{h}^{ik} = \delta_j^k \tag{3.29a}$$

Theorem 3.11. The following statements hold in X_m:

(a) The tensor $\bar{h}^{ij}$ defined by (3.29a) is the induced tensor h^{ij} on X_m of $h^{\lambda\nu}$. Hence

$$h_{ij}h^{ik} = \delta_j^k \tag{3.29b}$$

[5]The same induced metric tensor g_{ij} may be obtained from (3.3) in the usual way.

[6]Since the metric of X_n is not assumed to be positive definite, it is possible that on certain subspaces there exist points for which $\mathrm{Det}(h_{ij}) = 0$. In order to avoid this possibility and the complications resulting therefrom, we impose . . .

(b) The tensors h^{ij} and h_{ij} may be used for raising and/or lowering indices of the induced tensors on X_m in the usual manner.

Proof. In virtue of (3.26), (3.24), and (3.15)-(3.17), we have

$$\begin{aligned} h_{ij}h^{ik} &= (h_{\alpha\beta}B_i^\alpha B_j^\beta)(h^{\gamma\varepsilon}B_\gamma^i B_\varepsilon^k) \\ &= h_{\alpha\beta}h^{\gamma\varepsilon}B_j^\beta B_\varepsilon^k B_\gamma^\alpha \\ &= \delta_j^k - \sum_x (\underset{x}{N}_\beta B_j^\beta)(\overset{x}{N}{}^\varepsilon B_\varepsilon^k) = \delta_j^k \end{aligned}$$

which together with (3.29) proves statement (a). For the sake of simplicity of the proof of statement (b), consider a tensor $T_{\lambda\mu}$ in X_n and its induced tensor T_{ij} on X_m. Then in virtue of (3.8) and (3.18), we have

$$T_{ij}h^{jk} = T_{\alpha\beta}B_i^\alpha B_j^\beta h^{jk} = T_{\alpha\beta}B_i^\alpha h^{\beta\gamma}B_\gamma^k = T_\alpha^\gamma B_i^\alpha B_\gamma^k = T_i^k$$

which proves statement (b). ■

It should be noted, however, that the reverse relations of (3.26) are given, as in the following forms in virtue of (3.9):

$$h_{\lambda\mu} = h_{ij}B_\lambda^i B_\mu^j + \sum_x \varepsilon_x \overset{x}{N}_\lambda \overset{x}{N}_\mu \tag{3.30a}$$

$$h^{\lambda\nu} = h^{ij}B_i^\lambda B_j^\nu + \sum_x \varepsilon_x \underset{x}{N}{}^\lambda \underset{x}{N}{}^\nu \tag{3.30b}$$

3.3. Induced Connections

Definition 3.12. If $\Gamma_{\lambda\mu}^\nu$ is a connection on X_n, the connection Γ_{ij}^k defined by

$$\Gamma_{ij}^k = B_\gamma^k(B_{ij}^\gamma + \Gamma_{\alpha\beta}^\gamma B_i^\alpha B_j^\beta) \tag{3.31}$$

where

$$B_{ij}^\alpha = \frac{\partial B_i^\alpha}{\partial x^j} = \frac{\partial^2 y^\alpha}{\partial x^i\,\partial x^j} \tag{3.32}$$

is called *the induced connection on X_m derived from $\Gamma_{\lambda\mu}^\nu$ on X_n.*

It should be remarked that the torsion tensor $S_{ij}{}^k$ of the induced connection Γ_{ij}^k is the induced tensor on X_m of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma_{\lambda\mu}^\nu$ in X_n. That is,

$$S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B_i^\alpha B_j^\beta B_\gamma^k \tag{3.33}$$

Theorem 3.13. The induced connection $\{{}^{k}_{ij}\}$ on X_m derived from the Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$ on X_n are the Christoffel symbols defined by h_{ij}. That is,

$$\{{}^{k}_{ij}\} = \tfrac{1}{2}h^{kp}(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij}) \tag{3.34}$$

Proof. In virtue of (3.18), (3.24), and (3.31), our assertion (3.34) follows in the following way:

$$\begin{aligned}&\tfrac{1}{2}h^{kp}(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij})\\ &\quad = h_{\alpha\beta}(h^{pk}B^{\beta}_{p})B^{\alpha}_{ij} + \tfrac{1}{2}(h^{pk}B^{\varepsilon}_{p})(\partial_\beta h_{\alpha\varepsilon} + \partial_\alpha h_{\beta\varepsilon} - \partial_\varepsilon h_{\alpha\beta})B^{\alpha}_{i}B^{\beta}_{j}\\ &\quad = B^{k}_{\gamma}(\{{}^{\gamma}_{\alpha\beta}\}B^{\alpha}_{i}B^{\beta}_{j} + B^{\gamma}_{ij}) = \{{}^{k}_{ij}\}\end{aligned}$$

■

3.4. The Tensors $\overset{x}{\Omega}_{ij}$

In this section we introduce the tensors $\overset{x}{\Omega}_{ij}$, called *the generalized coefficients of the second fundamental form of* X_m. Let $\overset{0}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to the x's. Then

$$\overset{0}{D}_j B^{\alpha}_{i} = B^{\alpha}_{ij} + \Gamma^{\alpha}_{\beta\gamma}B^{\beta}_{i}B^{\gamma}_{j} - \Gamma^{k}_{ij}B^{\alpha}_{k} \tag{3.35}$$

Theorem 3.14. The vector $\overset{0}{D}_j B^{\nu}_{i}$ in X_n is normal to X_m and may be given by

$$\overset{0}{D}_j B^{\alpha}_{i} = -\sum_{x} \overset{x}{\Omega}_{ij} \underset{x}{N}^{\alpha} \tag{3.36}$$

where

$$\overset{x}{\Omega}_{ij} = -(\overset{0}{D}_j B^{\alpha}_{i})\, \overset{x}{N}_{\alpha} \tag{3.37}$$

Proof. Multiplying by B^{m}_{α} on both sides of (3.35), we have $(\overset{0}{D}_j B^{\alpha}_{i})B^{m}_{\alpha} = 0$, which shows that the vector $\overset{0}{D}_j B^{\nu}_{i}$ is normal to X_m. The relation (3.37) follows from (3.36) by multiplying by $\overset{y}{N}_{\alpha}$ on both sides of (3.36) and making use of (3.16). ■

Theorem 3.15. The tensors $\overset{x}{\Omega}_{ij}$ are the induced tensors on X_m of the tensor $D_\beta N^{\alpha}$ in X_n. That is,

$$\overset{x}{\Omega}_{ij} = (D_\beta \overset{x}{N}_{\alpha})B^{\alpha}_{i}B^{\beta}_{j} \tag{3.38}$$

Proof. Substituting (3.35) into (3.37) and making use of (3.16) and the relation

$$0=\partial_j(B_i^\alpha \overset{x}{N}_\alpha)=B_{ij}^\alpha \overset{x}{N}_\alpha+(\partial_\beta \overset{x}{N}_\alpha)B_i^\alpha B_j^\beta$$

we may easily derive the relation (3.38). ■

4. THE SUBMANIFOLDS OF SEX$_n$ AND THE SE IDENTITY

In this section, we shall prove that the induced connection on a submanifold of SEX$_n$ is the SE connection and that a very useful so-called SE identity holds on a submanifold of SEX$_n$.

Theorem 4.1. The induced connection Γ_{ij}^k on a subspace X_m of the SE connection $\Gamma_{\lambda\mu}^\nu$ on X_n is of the form

$$\Gamma_{ij}^k=\{{}_{ij}^{k}\}+2k_{(i}{}^kX_{j)}+2\delta_{[i}{}^kX_{j]} \tag{4.1}$$

where $\{{}_{ij}^{k}\}$ are the induced Christoffel symbols defined by (3.34) and X_i is the induced vector on X_m of the unique vector X_λ determining $\Gamma_{\lambda\mu}^\nu$. That is,

$$X_i=X_\alpha B_i^\alpha \tag{4.2}$$

Proof. Substituting (2.12) into (3.31) and making use of (3.24) and (4.2), we have (4.1). ■

Theorem 4.2. The induced connection Γ_{ij}^k on X_m, given by (4.1), of $\Gamma_{\lambda\mu}^\nu$ on X_n is an SE connection.

Proof. In virtue of (4.1), it is obvious that Γ_{ij}^k is semisymmetric. In order to prove that Γ_{ij}^k is Einstein, introduce the symbols D_j to denote *the symbolic vector of the covariant derivative with respect to* Γ_{ij}^k. In virtue of (2.8b), (2.9), (3.25), and (4.2), it follows from (3.24) that

$$\begin{aligned}
D_k g_{ij} &= (D_\omega g_{\lambda\mu})B_k^\omega B_i^\lambda B_j^\mu \\
&= 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}B_k^\omega B_i^\lambda B_j^\mu \\
&= 2(\delta_\omega^\alpha X_\mu g_{\lambda\alpha}-\delta_\mu^\alpha X_\omega g_{\lambda\alpha})B_k^\omega B_i^\lambda B_j^\mu \\
&= 2(X_\mu g_{\lambda\omega}-X_\omega g_{\lambda\mu})B_k^\omega B_i^\lambda B_j^\mu \\
&= 2(X_j g_{ik}-X_k g_{ij}) \\
&= 4\delta_{[k}{}^pX_{j]}g_{ip} \\
&= 2S_{kj}{}^p g_{ip}
\end{aligned}$$

which shows that Γ_{ij}^k satisfies the Einstein condition. ■

Remark 4.3. In virtue of the above theorem, we note that every subspace of SEX$_n$ connected by the induced connection of the SE connection on a SEX$_n$ is also an SE manifold.

Agreement 4.4. In our further considerations, *we use the symbol X_m exclusively to denote that submanifold of SEX_n connected by the induced connection Γ^k_{ij} of $\Gamma^\nu_{\lambda\mu}$ on SEX_n.*

Our final consequence in this section is the discovery of the following useful identity, which holds on X_m. Indeed, this identity will give many applications and contributions in the future study of the structure of X_m.

Theorem 4.5. (The SE identity.) On X_m the following identity holds:

$$\sum_x k_{\alpha\beta}(\overset{x}{\Omega}_{ik}B^\alpha_j - \overset{x}{\Omega}_{kj}B^\alpha_i)\underset{x}{N}^\beta = 0 \tag{4.3}$$

Proof. Since the induced connection Γ^k_{ij} is Einstein in virtue of Theorem 4.2, it must satisfy the following Einstein equations:

$$\partial_k g_{ij} - \Gamma^p_{ik} g_{pj} - \Gamma^p_{kj} g_{ip} = 0 \tag{4.4}$$

In virtue of (3.35) and (3.36), we first note that

$$B^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma} B^\beta_i B^\gamma_j = \Gamma^k_{ij} B^\alpha_k - \sum_y \overset{y}{\Omega}_{ij}\underset{y}{N}^\alpha \tag{4.5}$$

Substituting (3.25) and (3.31) into (4.4) and making use of (3.15), we have

$$\partial_k(g_{\alpha\beta}B^\alpha_i B^\beta_j) - (B^\gamma_{ik} + \Gamma^\gamma_{\alpha\beta}B^\alpha_i B^\beta_k)g_{\lambda\mu}B^\mu_j B^\lambda_\gamma$$
$$- (B^\gamma_{kj} + \Gamma^\gamma_{\alpha\beta}B^\alpha_k B^\beta_j)g_{\lambda\mu}B^\lambda_i B^\mu_\gamma = 0 \tag{4.6}$$

The first term of (4.6) may be rewritten as

$$\text{First Term} = (\partial_\gamma g_{\alpha\beta})B^\alpha_i B^\beta_j B^\gamma_k + g_{\alpha\beta}B^\alpha_{ik}B^\beta_j + g_{\alpha\beta}B^\alpha_i B^\beta_{jk} \tag{4.7}$$

In virtue of (3.19a), the second term of (4.6) can be written as

Second Term

$$= -(B^\gamma_{ik} + \Gamma^\gamma_{\alpha\beta}B^\alpha_i B^\beta_k)g_{\lambda\mu}B^\mu_j(\delta^\lambda_\gamma - \sum_x \underset{x}{N}^\lambda \overset{x}{N}_\gamma)$$

$$= -(B^\gamma_{ik} + \Gamma^\gamma_{\alpha\beta}B^\alpha_i B^\beta_k)g_{\gamma\mu}B^\mu_j$$

$$+ (\Gamma^p_{ik}B^\gamma_p - \sum_y \overset{y}{\Omega}_{ik}\underset{y}{N}^\gamma)g_{\lambda\mu}B^\mu_j \sum_x \underset{x}{N}^\lambda \overset{x}{N}_\gamma$$

$$= -g_{\alpha\beta}B^\alpha_{ik}B^\beta_j - g_{\epsilon\beta}\Gamma^\epsilon_{\alpha\gamma}B^\alpha_i B^\beta_j B^\gamma_k - g_{\lambda\mu}B^\mu_j \sum_x \overset{x}{\Omega}_{ik}\underset{x}{N}^\lambda \tag{4.8a}$$

where use has been made of (3.16) and (4.5). But the relations (2.2), (3.16), and (3.17b) allow the third term of the last equality of (4.8a) to be expressed in the form

$$-g_{\lambda\mu}B_j^{\mu}\sum_x \overset{x}{\Omega}_{ik}\underset{x}{N}^{\lambda} = k_{\alpha\beta}\sum_x \overset{x}{\Omega}_{ik}B_j^{\alpha}\underset{x}{N}^{\beta}$$

Substituting this into (4.8a), we have

Second Term

$$= -g_{\alpha\beta}B_{ik}^{\alpha}B_j^{\beta} - g_{\varepsilon\beta}\Gamma_{\alpha\gamma}^{\varepsilon}B_i^{\alpha}B_j^{\beta}B_k^{\gamma} + k_{\alpha\beta}\sum_x \overset{x}{\Omega}_{ik}B_j^{\alpha}\underset{x}{N}^{\beta} \tag{4.8b}$$

Similarly, the third term of (4.6) may be obtained as

Third Term

$$= -g_{\alpha\beta}B_i^{\alpha}B_{kj}^{\beta} - g_{\alpha\varepsilon}\Gamma_{\gamma\beta}^{\varepsilon}B_i^{\alpha}B_j^{\beta}B_k^{\gamma} + k_{\alpha\beta}\sum_x \overset{x}{\Omega}_{kj}B_i^{\alpha}\underset{x}{N}^{\beta} \tag{4.9}$$

We now substitute (4.7), (4.8b), and (4.9) into (4.6) to find

$$(\partial_{\gamma}g_{\alpha\beta} - g_{\varepsilon\beta}\Gamma_{\alpha\gamma}^{\varepsilon} - g_{\alpha\varepsilon}\Gamma_{\gamma\beta}^{\varepsilon})B_i^{\alpha}B_j^{\beta}B_k^{\gamma} + \sum_x k_{\alpha\beta}(\overset{x}{\Omega}_{ik}B_j^{\alpha} - \overset{x}{\Omega}_{kj}B_i^{\alpha})\underset{x}{N}^{\beta} = 0$$

This proves (4.3) in virtue of (2.8a). ■

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